Problem Solving and Problem Posing in Geometry with Dynamic Software

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All good mathematics teachers emphasize problem solving in their classes. Students need to be exposed to a variety of problems as well as a number of different strategies to solve them (NCTM, 1989). Unfortunately, solving problems is not enough; students also need the opportunities to formulate their own problems. This process of generating new problems and reformulating old problems is referred to as problem posing and it can occur prior to, during, and after problem solving (Silver, 1994). Problem solving and problem posing are complementary activities in mathematics education. In other words, one cannot solve problems unless there are problems stated or conjectured. Furthermore, the process of posing a problem can often provide more insight into the mathematics involved.

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Recent documents in mathematics education call for an increased attention to problem posing both as a curricular topic and as a pedagogical strategy. For instance, the *Principles and Standards for School Mathematics* (NCTM, 2000) states that teachers "should regularly ask students to formulate interesting problems based on a wide variety of situations, both within and outside mathematics" (p. 258). In the same vein, the document also recommends that students "should have opportunities to formulate and refine problems because problems that occur in real settings do not often arrive neatly packaged" (p. 335).

Technology can play an important role in problem posing. Students who use technology can "examine more examples or representational forms than are feasible by hand, so they can make and explore conjectures easily" (NCTM, 2000, p. 25). This paper is an effort to illustrate how dynamic software can help teachers and students both in problem solving and in problem posing. We chose a geometric context to illustrate our ideas. In the discussion, we use Cabri Geometry II, but any geometric dynamic software can be used to perform the activities presented.

We used this geometric context in a professional development project for secondary mathematics teachers. Note that the context does not provide a definite goal to achieve. We have found that taking away the goal from problems provides students with opportunities to formulate interesting problems and discover exciting relationships (Martínez-Cruz & Contreras, submitted).

A Geometric Context for Problem Solving and Problem Posing

Construct an equilateral triangle ABC. On each side of the triangle, construct squares. Name them ADEB, CBFG, and ACHI respectively. Extend sides FG, HI and DE until they intersect. Label the intersection points J, K, and L. See Figure 1. Find as many relationships as you can in this figure.

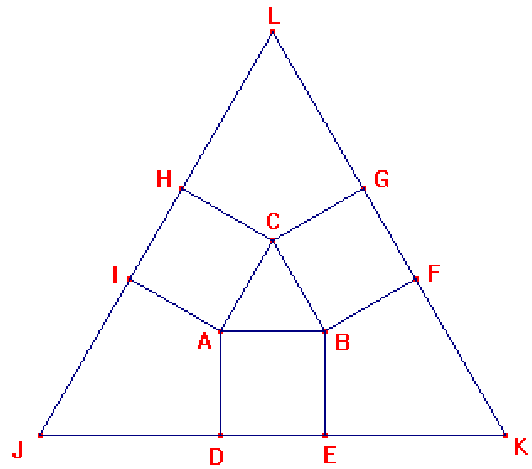

Figure 1

Since the problem is open-ended, it allows one to experiment in different directions. In this paper, we focus on ratios of areas. However, readers can observe that the proofs of these relationships involve various properties of geometric shapes. Hence the activity also provides opportunities to establish connections within mathematics, another goal recommended for mathematics students (NCTM, 2000).

Ratios of Areas

Let's begin with a question. Is there a relationship between some of the areas of the shapes formed in this construction? If so, which one can you establish? Once the question is raised, students can choose the figures they prefer. We start with triangles ABC and JKL, and square ACHI. Figure 2 shows a particular instance of the areas of triangles ABC and JKL, and square ACHI. Additionally, we computed three ratios: $\frac{Area\ \Delta JKL}{Area\ \Delta ABC} \approx 19.93$, $\frac{Area\ \Delta JKL}{Area\ square\ IACH} \approx 8.63$ and $\frac{Area\ square\ IACH}{Area\ \Delta ABC} \approx 2.31$. Each ratio is obtained first by calculating the areas needed and second using the Calculate command to compute the ratio.

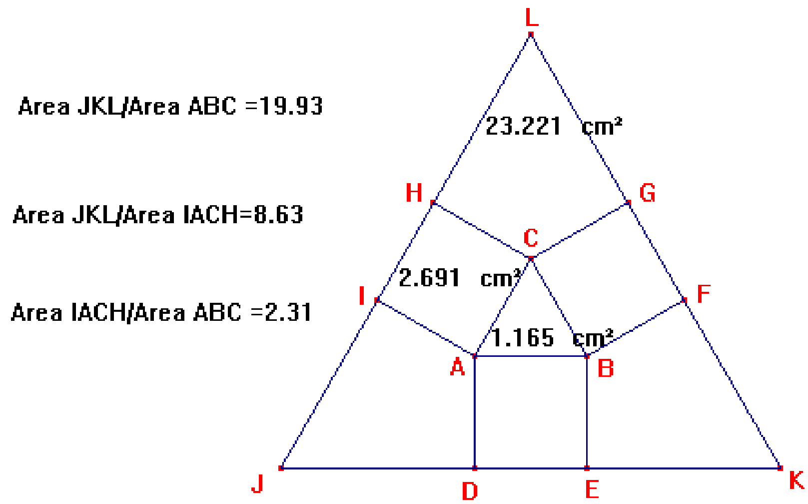

Figure 2

Next we use the software to explore a possible relationship between these areas. After dragging vertex A, the starting point in the construction, we observe that the ratios remain unchanged (Figure 3). This suggests that the areas might be related. So now we have a conjecture. We proceed to show that these ratios are constant and determine their exact value.

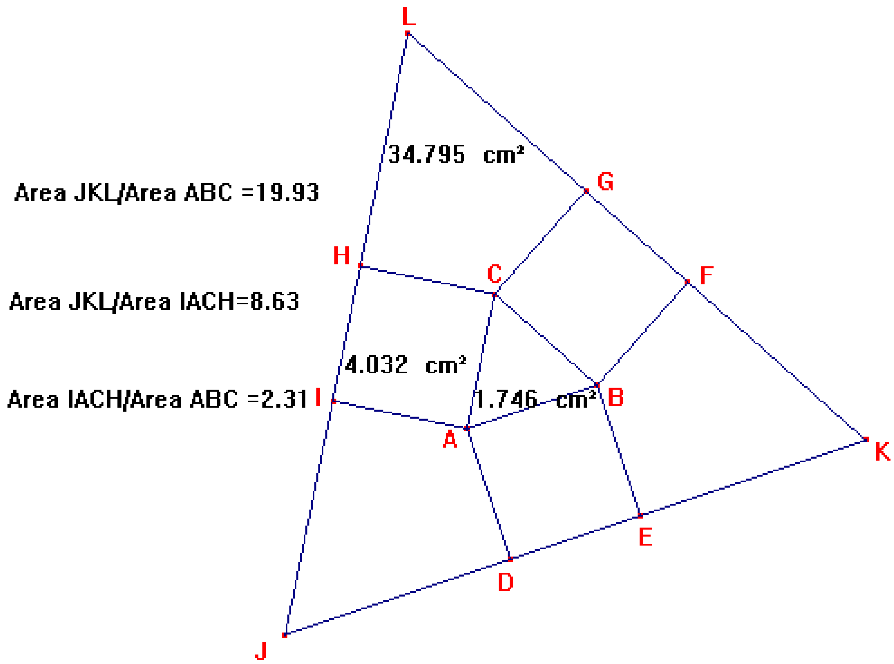

Figure 3

If $\overline{AB} = a$ then we have $height\ \Delta ABC = \frac{\sqrt{3}a}{2}$ (which is obtained using the Pythagorean Theorem), $Area\ \Delta ABC = \frac{\sqrt{3}a^2}{4}$ and $Area\ square\ ACHI = a^2$. Therefore, $\frac{Area\ square\ ACHI}{Area\ \Delta ABC} = \frac{4}{\sqrt{3}}$. To calculate the area of triangle JKL, we decompose it into smaller figures, namely three squares (congruent to square ADEB), triangle ABC, and three quadrilaterals (congruent to quadrilateral JDAI). We only need to calculate the area of quadrilateral JDAI. To do so, we construct segment AJ and obtain two $30^\circ - 90^\circ - 60^\circ$ congruent triangles, JDA and JIA. Each of these triangles has area $\frac{\sqrt{3}a^2}{2}$. Similarly, triangles CGL, CLH, KFB and BEK have the same area. Therefore the area of triangle JKL is $Area\ \Delta ABC + 3 * area\ square\ ACHI + 6 * area\ \Delta JDA = \frac{\sqrt{3}a^2}{4} + 3a^2 + 6\left(\frac{\sqrt{3}a^2}{2}\right)$

$= \frac{a^2}{4}\left(12 + 13\sqrt{3}\right)$. Hence $\frac{Area\ \Delta JKL}{Area\ square\ ACHI} = \frac{12 + 13\sqrt{3}}{4}$ and $\frac{Area\ \Delta JKL}{Area\ \Delta ABC} = \frac{12\sqrt{3} + 39}{3}$.

Furthermore, $\frac{Area\ \Delta JDA}{Area\ \Delta ABC} = 2$. Notice that the approximations provided by the dynamic software are close to the actual ratios.

Another problem to investigate is to construct segment $\overline{DI}$ and investigate other relationships (figure 4). First, we calculate $area\ \Delta IDA$ and $area\ \Delta IJD$. Computations with the software suggest the following relationships: $\dfrac{Area\ \Delta DAI}{Area\ \Delta ABC}=1$ and $\dfrac{Area\ JDI}{Area\ ABC}=3$. Moreover, measurements of angles in ΔJDI with the software suggest that ΔJDI is equilateral.

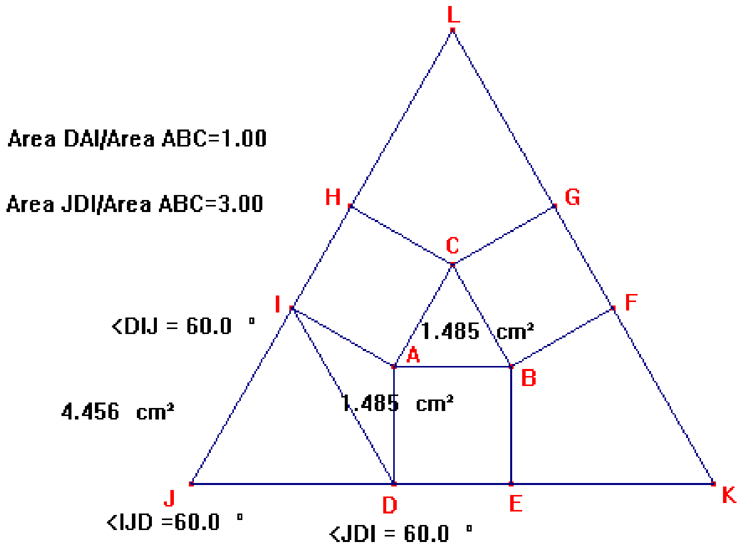

Fig. 4

We now prove that both conjectures are true. First, note that ΔDAI is isosceles and $\angle DAI = 120^{\circ}$. If $\overline{DI}$ is the base and h is the height of ΔDAI, we apply trigonometry to determine these values. The height, $h=\dfrac{a}{2}$, is found using $\sin 30^{\circ}=\dfrac{h}{a}$, while $\overline{DI}=\sqrt{3}a$, is obtained using the Pythagorean Theorem. Hence $area\ \Delta DAI=\dfrac{\sqrt{3}a^2}{4}$, which coincides with $Area\ \Delta ABC$. Therefore $\dfrac{Area\ \Delta DAI}{Area\ ABC}=1$. We now prove that ΔIJD is equilateral. We just showed that $\angle AID = 30^{\circ}$. Therefore $\angle DIJ = 60^{\circ}$. Similarly, $\angle JDI = 60^{\circ}$ and since the sum of interior angles in a triangle is 180°, we get $\angle IJD = 60^{\circ}$. So ΔDIJ is equilateral. We now find $area\ \Delta JDI$. To determine its height, we use similarity with ΔABC and find $height\ \Delta JDI=\dfrac{3a}{2}$. Since $\overline{DI}=\sqrt{3}a$ we obtain $area\ \Delta JDI=\dfrac{3\sqrt{3}a^2}{4}$, which gives $\dfrac{Area\ \Delta JDI}{Area\ \Delta ABC}=3$. We can also calculate $\dfrac{area\ \Delta JKL}{area\ \Delta JDI}=\dfrac{\frac{a^2}{4}\left(12+13\sqrt{3}\right)}{\frac{3\sqrt{3}a^2}{4}}=\dfrac{12\sqrt{3}+39}{9}$. Notice that since ΔDIJ is equilateral, we can also conclude that ΔJKL is equilateral.

Finally, let's compare $Area\ \Delta JKL$ and $Area\ hexagon\ HIDEFG$. $Area\ hexagon\ HIDEFG$ is obtained by adding the area of three squares (congruent to square ADEB), area ΔABC and the area of three triangles (congruent to ΔDAI). Therefore, $Area\ hexagon\ HIDEFG = 3a^2 + \frac{\sqrt{3}a^2}{4} + 3 * \frac{\sqrt{3}a^2}{4} = 3a^2 + \sqrt{3}a^2 = a^2(3+\sqrt{3})$ and from here

$$\frac{Area\ \Delta JKL}{Area\ hexagon\ HIDEFG} = \frac{\frac{a^2}{4}\left(12+13\sqrt{3}\right)}{a^2\left(3+\sqrt{3}\right)} = \frac{9\sqrt{3}-1}{8}.$$

Conclusions

In this paper, we have attempted to illustrate how mathematical explorations and conjecture making can be enhanced with technological tools. In particular, we used dynamic software to discover some relationships between the areas of various shapes in a geometric context. Once the relationships were conjectured, we moved on to applying a central tool of mathematics, that of a proof, to validate the conjectures. The proofs also provide opportunities to apply and integrate several mathematical concepts. The activity can be used for several purposes: problem solving, problem posing, and connections within mathematics.

Technology played an essential role in conjecture making. Many of the explorations were facilitated and motivated by the dynamic software. No wonder it has been said that "technology not only influences how mathematics is taught and learned but also affects what is taught and when a topic appears in the curriculum," (NCTM, 2000, p. 26).

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We close the paper with an invitation to use dynamic software to investigate possible relationships with other geometric concepts such as perimeter, parallel lines, congruence, or similarity. Even if one finds no relationships, there is always the reward of developing an inquisitive disposition.

References

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CORRECTION: We inadvertently did not include the articles of Michael Krach and Jennifer Holloman on the table of contents page of the 2000 Fall Issue of the *Ohio Journal of School Mathematics*. We sincerely apologize. Eds.